Enhancing the role of column representatives in testing the invariance properties of switching systems

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***Abstract*—The role played by column representatives is expanded in the study of the properties exhibited by the trajectories of arbitrary switching linear systems. Previous contributions on the employment of column representatives focused on positive dynamics and linear copositive Lyapunov functions associated with exponentially contractive positive sets that are invariant with respect to such dynamics. Our approach refers to arbitrary dynamics and invariant sets with general form for time-dependence. We address both discrete- and continuous-time cases. Our key finding is that the existence of such invariant sets is fully characterized (if and only if) by the Schur (Hurwitz respectively) stability of the column representatives corresponding to a matrix set adequately built from the original system matrices. Our mathematical developments are illustrated by a numerical example. These developments incorporate the previous contributions mentioned above as particular cases.**

Keywords—Switching systems, column representatives, time-dependent invariant sets.

I. INTRODUCTION

A. Research framework

The paper develops a deeper insight into the properties of the trajectories of arbitrary switching linear systems, by enhancing the role of an algebraic instrument based on matrix representatives. The considered dynamics are described, in discrete-time, by

$$x(t+1)=A_{\upsilon(t)}x(t),\ x(t_0)=x_0,\quad A_{\upsilon(t)}\in\mathcal{A},\ \ t,t_0\in\mathbb{Z}_+,\ t\ge t_0, \tag{1-DT}$$

and, in continuous-time, by

$$\dot{x}(t)=A_{\upsilon(t)}x(t),\ x(t_0)=x_0,\quad A_{\upsilon(t)}\in\mathcal{A},\ \ t,t_0\in\mathbb{R}_+,\ t\ge t_0, \tag{1-CT}$$

where the concrete values of the subscripts $\upsilon(t)$ are given by an arbitrarily switching signal $\upsilon:\mathbb{Z}_+\to\{1,2,\dots,N\}$ (discrete-time), respectively $\upsilon:\mathbb{R}_+\to\{1,2,...,N\}$ (continuous-time). At any moment *t*, the value $\theta=\upsilon(t)$ selects a matrix from the set (called the set of *constituent matrices* of the switching system)

$$\mathcal{A}=\{A_1,A_2,...,A_N\}\subset\mathbb{R}^{n\times n} \tag{2}$$

and activates the *constituent subsystem* corresponding to $\theta=\upsilon(t)$, meaning

$$x(t+1)=A_\theta x(t),\quad t\in\mathbb{Z}_+,\ \theta\in\{1,\dots,N\} \tag{3-DT}$$

for discrete-time dynamics, and

$$\dot{x}(t)=A_\theta x(t),\quad t\in\mathbb{R}_+,\ \theta\in\{1,\dots,N\} \tag{3-CT}$$

for discrete-time dynamics.

In the equation labels, we have introduced the notation “DT”, “CT” respectively, as extension for numbering; this is an abbreviation for “discrete-time”, and “continuous-time” respectively. Our text will preserve the meaning of this notation for superscripts attached to some matrices. We will also write “X // Y” in place of "X [respectively Y]", aiming to a parallel presentation of several statements that refer to similar aspects encountered in discrete- and continuous-time dynamics.

The *positive case* of arbitrary switching systems (1-DT) // (1-CT) has been thoroughly studied by separate works, entirely devoted to this case, such as [1]-[7]. In these works, the constituent matrices in (2) are *non-negative* for discrete-time systems and *essentially non-negative* for continuous-time systems. One of the key instruments used by these approaches is the set of *column representatives* associated with the constituent matrices (2), as briefly presented below. For any function $\sigma:\{1,\dots,n\}\to\{1,\dots,N\}$, consider $\boldsymbol{\sigma}=(\sigma(1),\dots,\sigma(n))$ the ordered *n*-tuple and denote by $\mathcal{C}$ the set of all the *n*-tuples with elements from $\{1,\dots,N\}$. For any $\boldsymbol{\sigma}\in\mathcal{C}$, matrix

$$\underline{A}_{\boldsymbol{\sigma}}=\left[[A_{\sigma(1)}]_{(:,1)}\ \dots\ [A_{\sigma(n)}]_{(:,n)}\right]\in\mathbb{R}^{n\times n}, \tag{4}$$

is built column-wise from the first column of $A_{\sigma(1)}$ (denoted by $[A_{\sigma(1)}]_{(:,1)}$), the second column of $A_{\sigma(2)}$ (denoted by $[A_{\sigma(2)}]_{(:,2)}$), and so on. Matrix $\underline{A}_{\boldsymbol{\sigma}}\in\mathbb{R}^{n\times n}$ is called a *column representative* of the matrix set $\mathcal{A}$.

B. Existent results employing the column representatives of switching positive systems

Papers [5] // [1]-[3] show that the Schur // Hurwitz stability of all column representatives of the discrete-time system (1-DT) // continuous-time system (1-CT) represents a *necessary and sufficient condition* for the existence of *linear copositive Lyapunov functions*

$$U:\mathbb{R}^n_+\to\mathbb{R}_+,\ U(x)=u^Tx,\quad u=[u_1\dots u_n]\in\mathbb{R}^n,\ u_i>0,\ i=1,\dots,n, \tag{5}$$

associated with system (1-DT) // (1-CT). The positive vector $u\gg 0$ used in (5) can be computed as a solution to the strong linear inequalities

$$u^T(A_\theta - I) \ll 0,\ \theta = 1,\dots,N, \quad \text{(6-DT)}$$

in the discrete-time case and

$$u^T A_\theta \ll 0,\ \theta = 1,\dots,N, \quad \text{(6-CT)}$$

in the continuous-time case (where the meaning of the notations "$\gg$", "$\ll$" is in accordance with Section II).

Our papers [4], [6] and [8] expanded the framework proposed by the above mentioned articles, by proving the equivalence of the following four statements

(S1) All column representatives $\underline{A}_{\sigma}$, $\sigma \in \mathcal{C}$, corresponding to the set of constituent matrices of system (1-DT) // (1-CT) are Schur // Hurwitz stable.

(S2) The quasi–linear weak inequalities are solvable

$$u^T A_\theta \le r u^T,\ 0 < r < 1,\ u \gg 0,\ \theta = 1,\dots,N \quad \text{(7-DT)}$$

$$u^T A_\theta \le r u^T,\ r < 0,\ u \gg 0,\ \theta = 1,\dots,N \quad \text{(7-CT)}$$

(S3) There exist *linear copositive Lyapunov functions* of form (5) with the decrease rate r along each non-trivial trajectory of system (1-DT) // (1-CT)

$$U(x(t+1)) \le rU(x(t)),\ t \in \mathbb{Z}_+, u \gg 0,\ 0 < r < 1, \quad \text{(8-DT)}$$

$$D_t^+ U(x(t)) = \lim_{\tau \downarrow 0} \frac{U(x(t+\tau)) - U(x(t))}{\tau} \le rU(x(t)), \quad \text{(8-CT)}$$
$$t \in \mathbb{R}_+, u \gg 0,\ r < 0$$

(S4) There exist sets with exponentially-contractive form

$$X^{\mathrm{DT}}(t) = \left\{ x \in \mathbb{R}_+^n \mid u^T x \le r^t \right\}, \quad \text{(9-DT)}$$
$$t \in \mathbb{Z}_+, u \gg 0,\ 0 < r < 1,$$

$$X^{\mathrm{CT}}(t) = \left\{ x \in \mathbb{R}_+^n \mid u^T x \le e^{rt} \right\}, \quad \text{(9-CT)}$$
$$t \in \mathbb{R}_+,\ u \gg 0,\ r < 0,$$

which are invariant with respect to the trajectories of system (1-DT) // (1-CT).

C. Objectives of the current research

The objective of the current paper is to enhance the role of the column representatives in exploring the trajectory properties for switching systems (1-DT) // (1-CT). The scenario briefly described by subsection 1.2 is extended by considering the following less conservative hypotheses for the dynamics:

- The switching system (1-DT) // (1-CT) may be defined by *arbitrary constituent matrices* (2) (i.e. not only nonnegative // essentially nonnegative).
- The candidates for invariant sets may have *arbitrary time dependence* (i.e. not only exponentially-contractive form)

$$X_\Gamma^{\mathrm{DT}}(t) = \left\{ x \in \mathbb{R}^n \mid \|\Gamma(t)x\|_1 \le 1 \right\},\ t \in \mathbb{Z}_+,$$
$$\Gamma(t) = \mathrm{diag}\{\gamma_1(t),\dots,\gamma_n(t)\}, \quad \text{(10-DT)}$$
$$\gamma_i(t) > 0, \forall t \in \mathbb{Z}_+,\ \lim_{t\to\infty} \gamma_i(t) = \infty,\ i = 1,\dots,n,$$

$$X_\Gamma^{\mathrm{CT}}(t) = \{ x \in \mathbb{R}^n \mid \|\Gamma(t)x\|_1 \le 1 \},\ t \in \mathbb{R}_+,$$
$$\Gamma(t) = \mathrm{diag}\{\gamma_1(t),\dots,\gamma_n(t)\}, \quad \text{(10-CT)}$$
$$\gamma_i(t) > 0, \forall t \in \mathbb{R}_+,\ \lim_{t\to\infty} \gamma_i(t) = \infty,\ i = 1,\dots,n,$$

The main contribution consists in proving that the invariance of sets (10-DT) // (10-CT) can be characterized, by equivalence, via the column representatives of a set of matrices, which are adequately built from the constituent matrices (2). The results are separately presented for discrete-time and continuous-time dynamics. We also show that the equivalence between (S1), …, (S4) can be obtained as a particular case of the new results, in both discrete-time and continuous-time cases.

Our exposition is organized in the following sections. Section II presents the notations and nomenclature used throughout the text. Section III and IV develop new results for discrete-, and, respectively continuous-time arbitrary switching linear systems. Section V illustrates the theoretical results corresponding to the continuous-time case by a numerical example. Section VI provides some final comments on the mathematical significance of the new developments.

II. Notations and Nomenclature

***(Essentially) nonnegative matrices and componentwise matrix inequalities* [9]**

A rectangular matrix $X = [x_{ij}] \in \mathbb{R}^{n\times m}$ is called: • *nonnegative* (notation $X \ge 0$) if $x_{ij} \ge 0$, $i = 1,\dots,n$, $j = 1,\dots,m$; • *positive* (notation $X \gg 0$) if $x_{ij} > 0$, $i = 1,\dots,n$, $j = 1,\dots,m$. A square matrix $X = [x_{ij}] \in \mathbb{R}^{n\times n}$ is called *essentially* nonnegative (positive) if $x_{ij} \ge 0$, ($x_{ij} > 0$), $i, j = 1,\dots,n$, $i \ne j$.

If $X, Y \in \mathbb{R}^{n\times m}$, then the componentwise inequality $X \ge Y$ ($X \gg Y$) means $X - Y \ge 0$ ($X - Y \gg 0$).

***Use of 1-norm* [9]**

Given $x = [x_1 \cdots x_n]^T \in \mathbb{R}^n$ and $M = [m_{ij}] \in \mathbb{R}^{n\times n}$, the following notations are used.

• *vector norm*: $\|x\|_1 = \sum_{i=1}^{n} |x_i|$; • *induced matrix-norm*:

$$\|M\|_1 = \sup_{x \ne 0} \frac{\|Mx\|_1}{\|x\|_1} = \max_{1\le j\le n} \left\{ \sum_{i=1}^{n} |m_{ij}| \right\};$$

• *matrix measure* corresponding to the matrix norm:

$$\mu_1(M) = \lim_{\tau\downarrow 0} \frac{\|I + \tau M\|_1 - 1}{\tau} = \max_{1\le j\le n} \left\{ m_{jj} + \sum_{i=1, i\ne j}^{n} |m_{ij}| \right\}.$$

***Eigenstructure of (essentially) nonnegative matrices* [10]**

Let $M \in \mathbb{R}^{n\times n}$ be (essentially) nonnegative and denote its eigenvalues by $\lambda_i(M)$, $i = 1,\dots,n$,. • If M is nonnegative then it has a real eigenvalue $\lambda_{PF}(M)$ satisfying $|\lambda_i(M)| \le \lambda_{PF}(M)$, $i = 1,...,n$. • If M is essentially nonnegative the it has a real eigenvalue $\lambda_{PF}(M)$ satisfying $\mathrm{Re}\{\lambda_i(M)\} \le \lambda_{PF}(M)$, $i = 1,...,n$. • The (essentially) nonnegative matrix M has a nonnegative right eigenvector $w_R(M) > 0$, satisfying $\|w_R(M)\|_1 = 1$ and a nonnegative left eigenvector $w_L(M) > 0$, satisfying $\|w_L(M)\|_1 = 1$ that correspond to the eigenvalue $\lambda_{PF}(M)$. • If M is (essentially) nonnegative and *irreducible* (i.e. its associated graph is strongly connected) then $\lambda_{PF}(M)$ is a simple eigenvalue and its corresponding right and left eigenvectors are positive $\boldsymbol{w}_R(\boldsymbol{M}) \gg 0$, $\boldsymbol{w}_L(\boldsymbol{M}) \gg 0$. (Perron-Frobenius eigenstructure).

III. Results for Discrete-Time Dynamics

Starting from the constituent matrices $\mathcal{A} = \{\boldsymbol{A}_1, \boldsymbol{A}_2, ..., \boldsymbol{A}_N\}$ (2) that define the dynamics of the switching system (1-DT), let us build the set of non-negative matrices

$$\mathcal{A}^{\mathrm{DT}} = \left\{\boldsymbol{A}_1^{\mathrm{DT}}, \boldsymbol{A}_2^{\mathrm{DT}}, ..., \boldsymbol{A}_N^{\mathrm{DT}}\right\}$$
$$[\boldsymbol{A}_\theta^{\mathrm{DT}}]_{ij} = |[\boldsymbol{A}_\theta]_{ij}|, \; i,j = 1,\dots,n, \; \theta = 1,\dots,N \qquad \text{(11-DT)}$$

Consider the column representatives $\underline{\boldsymbol{A}}_{\boldsymbol{\sigma}}^{\mathrm{DT}}$, $\boldsymbol{\sigma} \in \mathcal{C}$, corresponding to the set of nonnegative matrices (11-DT).

Theorem 1-DT

There exist sets of form $X_{\boldsymbol{\Gamma}}^{\mathrm{DT}}(t)$ (10-DT) that are invariant with respect to the arbitrary switching system (1-DT) if and only if all column representatives $\underline{\boldsymbol{A}}_{\boldsymbol{\sigma}}^{\mathrm{DT}}$, $\boldsymbol{\sigma} \in \mathcal{C}$, of the non-negative matrices $\mathcal{A}^{\mathrm{DT}}$ (11-DT) are Schur stable.

Proof: *Necessity:* If the set $X_{\boldsymbol{\Gamma}}^{\mathrm{DT}}(t)$ (10-DT) is invariant with respect to system (1-DT), then it is invariant with respect to the trajectories of each subsystem $\boldsymbol{x}(t+1) = \boldsymbol{A}_\theta \boldsymbol{x}(t)$, $\theta = 1,\dots,N$. We consider the state-space vector transformation $\boldsymbol{y}(t) = \boldsymbol{\Gamma}(t)\boldsymbol{x}(t)$, which, from system (1-DT), leads to the system defined by the constituent subsystems

$$\boldsymbol{y}(t+1) = \left(\boldsymbol{\Gamma}(t+1)\boldsymbol{A}_\theta \boldsymbol{\Gamma}^{-1}(t)\right)\boldsymbol{y}(t), \; t \in \mathbb{Z}_+, \; \theta = 1,\dots,N. \qquad \text{(12-DT)}$$

The condition $\|\boldsymbol{\Gamma}(t)x\|_1 \le 1$ defining the set $X_{\boldsymbol{\Gamma}}^{\mathrm{DT}}(t)$ (10-DT) means the invariance of the set $\|\boldsymbol{y}\|_1 \le 1$ with respect to system (12-DT). This is equivalent to

$$\left\|\boldsymbol{\Gamma}(t+1)\boldsymbol{A}_\theta \boldsymbol{\Gamma}^{-1}(t)\right\|_1 \le 1, \; t \in \mathbb{Z}_+, \; \theta = 1,\dots,N. \quad \text{(13-DT)}$$

Let $\boldsymbol{y}$ solve subsystem (12-CT) and let $t \in \mathbb{Z}_+$. Set $\varepsilon = \|\boldsymbol{y}(t)\|_1$. If $\varepsilon > 0$, then $\overline{\boldsymbol{y}} = \varepsilon^{-1}\boldsymbol{y}$ also solves (12-DT) and satisfies $\|\overline{\boldsymbol{y}}(t)\|_1 = 1$. Since the set $\|\boldsymbol{y}\|_1 \le 1$ is positively invariant with respect to the considered subsystem (12-DT), we have $\|\overline{\boldsymbol{y}}(t+1)\|_1 \le 1$, or, equivalently $\|\varepsilon^{-1}\boldsymbol{y}(t+1)\|_1 \le 1$. Thus, for any $t \in \mathbb{Z}_+$, $\|\boldsymbol{y}(t+1)\|_1 \le \varepsilon = \|\boldsymbol{y}(t)\|_1$, i.e. the function $\|\boldsymbol{y}(t)\|_1$ is non-increasing along each trajectory of subsystem (12-DT). Introduce the notation $\boldsymbol{M}_\theta(t) = \boldsymbol{\Gamma}(t+1)\boldsymbol{A}_\theta \boldsymbol{\Gamma}^{-1}(t)$. For arbitrary $t \in \mathbb{Z}_+$, and arbitrary $\theta = 1,\dots,N$, there exists a vector $\boldsymbol{y}_0 \in \mathbb{R}^n$, $\|\boldsymbol{y}_0\|_1 = 1$, such that $\|\boldsymbol{M}_\theta(t)\|_1 = \|\boldsymbol{M}_\theta(t)\boldsymbol{y}_0\|_1$. If $\boldsymbol{y}(t) = \boldsymbol{y}_0$, then for $\boldsymbol{y}(t+1) = \boldsymbol{M}_\theta(t)\boldsymbol{y}(t)$ we have $\|\boldsymbol{y}(t+1)\|_1 \le \|\boldsymbol{y}(t)\|_1 = 1$, due to the non-increasing monotonicity of the function $\|\boldsymbol{y}(t)\|_1$ along any trajectory of subsystem (12-DT). Hence $\|\boldsymbol{M}_\theta(t)\|_1 = \|\boldsymbol{y}(t+1)\|_1 \le 1$ for any $t \in \mathbb{Z}_+$, meaning that inequalities (13-DT) are true.

By using $\boldsymbol{\Gamma}(t)$ as defined in (10-DT), inequalities (13-DT) are further equivalent to

$$(\boldsymbol{A}_\theta^{\mathrm{DT}})^T \boldsymbol{\gamma}(t+1) \le \boldsymbol{\gamma}(t), \; t \in \mathbb{Z}_+, \; \theta = 1,\dots,N, \quad \boldsymbol{\gamma}: \mathbb{Z}_+ \to \mathbb{R}^n, \boldsymbol{\gamma}(t) = [\gamma_1(t) \dots \gamma_n(t)]^T, \qquad \text{(14-DT)}$$

which show that the inequalities

$$(\underline{\boldsymbol{A}}_{\boldsymbol{\sigma}}^{\mathrm{DT}})^T \boldsymbol{\gamma}(t+1) \le \boldsymbol{\gamma}(t), \; t \in \mathbb{Z}_+, \boldsymbol{\sigma} \in \mathcal{C} \qquad \text{(15-DT)}$$

hold true for all column representatives $\underline{\boldsymbol{A}}_{\boldsymbol{\sigma}}^{\mathrm{DT}}$, $\boldsymbol{\sigma} \in \mathcal{C}$, of the set of nonnegative matrices $\mathcal{A}^{\mathrm{DT}}$ (11-DT).

Since $\lim_{t\to\infty} \gamma_i(t) = \infty$, $i = 1,\dots,n$, we can find a moment $\underline{t} \in \mathbb{Z}_+$ such that $\gamma(0) \le \varphi\gamma(\underline{t})$, $0 < \varphi < 1$, which, together with (15-DT) yield the inequalities

$$\left[(\underline{\boldsymbol{A}}_{\boldsymbol{\sigma}}^{\mathrm{DT}})^T\right]^{\underline{t}} \boldsymbol{\gamma}(\underline{t}) \le \boldsymbol{\gamma}(0) \le \varphi\gamma(\underline{t}), \quad \boldsymbol{\gamma}(\underline{t}) \gg 0, \; 0 < \varphi < 1, \; \boldsymbol{\sigma} \in \mathcal{C}. \qquad \text{(16-DT)}$$

For any $\boldsymbol{\sigma} \in \mathcal{C}$, the Perron-Frobenius eigenvalue of the non-negative matrix $\left[(\underline{\boldsymbol{A}}_{\boldsymbol{\sigma}}^{\mathrm{DT}})^T\right]^{\underline{t}}$ fulfills the inequality (as per [11, Corollary 8.1.29])

$$\lambda_{PF}\left(\left[(\underline{\boldsymbol{A}}_{\boldsymbol{\sigma}}^{\mathrm{DT}})^T\right]^{\underline{t}}\right) \le \varphi < 1, \; \boldsymbol{\sigma} \in \mathcal{C}, \qquad \text{(17-DT)}$$

proving that all column representatives $\underline{\boldsymbol{A}}_{\boldsymbol{\sigma}}^{\mathrm{DT}}$, $\boldsymbol{\sigma} \in \mathcal{C}$, of the non-negative matrices $\mathcal{A}^{\mathrm{DT}}$ (11-DT) are Schur stable.

Sufficiency: The switching *positive* system (1-DT) with *non-negative* constituent matrices $\mathcal{A}^{\mathrm{DT}}$ (11-DT) has invariant sets of form $X^{\mathrm{DT}}(t)$ (9-DT) as per the implication (S1)⇒(S4) discussed in Subsection 1.2. This means that the switching positive system (1-DT) with the constituent matrices $\mathcal{A}^{\mathrm{DT}}$ (11-DT) can serve as a *comparison system* (e.g. see [12]) for the switching system (1-CT) with *arbitrary* constituent matrices $\mathcal{A}$ (2). Therefore, the nonnegative set $X^{\mathrm{DT}}(t)$ (9-DT) is invariant with respect to the 1-norm of the trajectories of switching system (1-DT) with arbitrary constituent matrices $\mathcal{A}$ (2). Equivalently, the sets of the form $\left\{\boldsymbol{x} \in \mathbb{R}^n \mid \|\mathrm{diag}\{\boldsymbol{u}\}\boldsymbol{x}\|_1 \le r^t\right\}$, $t \in \mathbb{R}_+$, $\boldsymbol{u} \gg 0$, $0 < r < 1$, are invariant with respect to the trajectories of the switching system (1-DT) with arbitrary constituent matrices $\mathcal{A}$ (2). These invariant sets can also be described as $\left\{\boldsymbol{x} \in \mathbb{R}^n \mid \|(1/r)^t \mathrm{diag}\{\boldsymbol{u}\}\boldsymbol{x}\|_1 \le 1\right\}$. In other words, we have proven the existence of a set of form $X_{\boldsymbol{\Gamma}}^{\mathrm{DT}}(t)$ (10-DT), with $\gamma_i(t) = u_i(1/r)^t$, $u_i > 0$, $i = 1,\dots,n$, $0 < r < 1$, which is invariant with respect to switching system (1-DT) with arbitrary constituent matrices $\mathcal{A}$ (2). ■

Remark 1-DT

The proof of Sufficiency of Theorem 1-DT uses the simplest form for the invariant sets $X_{\boldsymbol{\Gamma}}^{\mathrm{DT}}(t)$ (10-DT). It is worth sketching a procedure for constructing invariant sets with a more general form. If all column representatives

$\underline{A}_{\boldsymbol{\sigma}}^{\text{DT}}$, $\boldsymbol{\sigma}\in\mathcal{C}$, of the set of non-negative matrices $\mathcal{A}^{\text{DT}}$ (11-DT) are Schur stable, then, for the quasi–linear weak inequalities (7-DT) written for $\boldsymbol{A}_{\theta}^{\text{DT}}$, we can find several solutions, as per implication (S1)⇒(S2) discussed in Subsection 1.2. For instance, consider the vectors $\boldsymbol{u}_k \gg 0$, and the constants $0 < r_k < 1$, $k=1,\ldots,K$, which solve inequalities (7-DT) written for $\boldsymbol{A}_{\theta}^{\text{DT}}$, and define the vector function

$$\boldsymbol{\gamma}(t)=\sum_{k=1}^{K} c_k \boldsymbol{u}_k (1/r_k)^t,\ c_k>0, k=1,\ldots,K. \quad \text{(18-DT)}$$

The function $\boldsymbol{\gamma}(t)$ (18-DT) is a solution to inequalities (14-DT) and $\boldsymbol{\Gamma}(t)=\operatorname{diag}\{\boldsymbol{\gamma}(t)\}$ satisfies inequalities (13-DT). Equivalently, the set $X_{\boldsymbol{\Gamma}}^{\text{DT}}(t)$ (10-DT) defined by $\boldsymbol{\Gamma}(t)=\operatorname{diag}\{\sum_{k=1}^{K} c_k\boldsymbol{u}_k (1/r_k)^t\}$ is invariant with respect to the trajectories of the switching system (1-DT) with *arbitrary* constituent matrices $\mathcal{A}$ (2). ■

Remark 2-DT

Theorem 1-DT includes the equivalence (S1)⇔(S4) discussed in Subsection 1.2 as a particular case. Indeed, Theorem 1-DT can be applied to a switching *positive* system (1-DT) with $\underline{A}_{\boldsymbol{\sigma}}^{\text{DT}}=\underline{A}_{\boldsymbol{\sigma}}$, $\boldsymbol{\sigma}\in\mathcal{C}$, and $\boldsymbol{\Gamma}(t)=\operatorname{diag}\{(1/r)^t\boldsymbol{u}\}$, $\boldsymbol{u}\gg 0$, $0<r<1$. ■

IV. RESULTS FOR CONTINUOUS-TIME DYNAMICS

Starting from the constituent matrices $\mathcal{A}=\{\boldsymbol{A}_1,\boldsymbol{A}_2,...,\boldsymbol{A}_N\}$ (2) that define the dynamics of switching system (1-CT), let us build the set of essentially non-negative matrices

$$\mathcal{A}^{\text{CT}}=\left\{\boldsymbol{A}_1^{\text{CT}},\boldsymbol{A}_2^{\text{CT}},...,\boldsymbol{A}_N^{\text{CT}}\right\}$$
$$[\boldsymbol{A}_{\theta}^{\text{CT}}]_{ii}=[\boldsymbol{A}_{\theta}]_{ii};\ [\boldsymbol{A}_{\theta}^{\text{CT}}]_{ij}=|[\boldsymbol{A}_{\theta}]_{ij}|, i\neq j; \quad \text{(11-CT)}$$
$$i,j=1,\ldots,n,\quad \theta=1,\ldots,N.$$

Consider the column representatives $\underline{A}_{\boldsymbol{\sigma}}^{\text{CT}}, \boldsymbol{\sigma}\in\mathcal{C}$ corresponding to the set of essentially nonnegative matrices (11-CT).

Theorem 1-CT

There exist sets of form $X_{\boldsymbol{\Gamma}}^{\text{CT}}(t)$ (10-CT) that are invariant with respect to arbitrary switching system (1-CT) if and only if all column representatives $\underline{A}_{\boldsymbol{\sigma}}^{\text{CT}}, \boldsymbol{\sigma}\in\mathcal{C}$ of the essentially non-negative matrices $\mathcal{A}^{\text{CT}}$ (11-CT) are Hurwitz stable.

Proof: *Necessity:* If the set $X_{\boldsymbol{\Gamma}}^{\text{CT}}(t)$ (10-CT) is invariant with respect to system (1-CT), then it is invariant with respect to the trajectories of each subsystem $\dot{\boldsymbol{x}}(t)=\boldsymbol{A}_{\theta}\boldsymbol{x}(t)$, $\theta=1,\ldots,N$. We consider the state-space vector transformation $\boldsymbol{y}(t)=\boldsymbol{\Gamma}(t)\boldsymbol{x}(t)$, which, from system (1-CT), leads to the system defined by the constituent subsystems

$$\dot{\boldsymbol{y}}(t)=\left(\dot{\boldsymbol{\Gamma}}(t)\boldsymbol{\Gamma}^{-1}(t)+\boldsymbol{\Gamma}(t)A_{\theta}\boldsymbol{\Gamma}^{-1}(t)\right)\boldsymbol{y}(t),\quad t\in\mathbb{R}_+,\ \theta=1,\ldots,N. \quad \text{(12-CT)}$$

The condition $\|\boldsymbol{\Gamma}(t)\boldsymbol{x}\|_1\leq 1$ defining the set $X_{\boldsymbol{\Gamma}}^{\text{CT}}(t)$ (10-CT) means the invariance of the set $\|\boldsymbol{y}\|_1\leq 1$ with respect to system (12-CT). This is equivalent to

$$\mu_1(\dot{\boldsymbol{\Gamma}}(t)\boldsymbol{\Gamma}^{-1}(t)+\boldsymbol{\Gamma}(t)A_{\theta}\boldsymbol{\Gamma}^{-1}(t))\leq 0,\quad t\in\mathbb{R}_+,\ \theta=1,\ldots,N. \quad \text{(13-CT)}$$

Indeed, let $\boldsymbol{\Psi}_{\theta}(t,t_0)$, $\theta=1,\ldots,N$, be the transition matrix of an arbitrary subsystem (12-CT), which allows writing $\boldsymbol{y}(t)=\boldsymbol{\Psi}_{\theta}(t,t_0)\boldsymbol{y}(t_0)$, $t\geq t_0$, $t,t_0\in\mathbb{R}_+$. By using the simplified notation $\boldsymbol{M}_{\theta}(t)=\dot{\boldsymbol{\Gamma}}(t)\boldsymbol{\Gamma}^{-1}(t)+\boldsymbol{\Gamma}(t)\boldsymbol{A}_{\theta}\boldsymbol{\Gamma}^{-1}(t)$, for the transition matrix we have

$$\begin{aligned}\boldsymbol{\Psi}_{\theta}(t+\tau,t)&=\boldsymbol{\Phi}_{\theta}(t+\tau)\boldsymbol{\Phi}_{\theta}^{-1}(t)=\\&=\left(\boldsymbol{\Phi}_{\theta}(t)+\tau\,\dot{\boldsymbol{\Phi}}_{\theta}(t)+\tau\,\boldsymbol{O}_{\theta}(\tau)\right)\boldsymbol{\Phi}_{\theta}^{-1}(t)=\\&=\boldsymbol{I}+\tau\boldsymbol{M}_{\theta}(t)+\tau\,\boldsymbol{O}_{\theta}(\tau)\,\boldsymbol{\Phi}_{\theta}^{-1}(t),\end{aligned}$$

where $\boldsymbol{\Phi}_{\theta}(t)$ denotes a fundamental matrix of the considered subsystem (12-CT), satisfying $\dot{\boldsymbol{\Phi}}_{\theta}(t)=\boldsymbol{M}_{\theta}(t)\boldsymbol{\Phi}_{\theta}(t)$, and $\lim_{\tau\downarrow 0}\boldsymbol{O}_{\theta}(\tau)=\boldsymbol{O}$. Thus

$$\begin{aligned}&\|\boldsymbol{I}+\tau\boldsymbol{M}_{\theta}(t)\|_1-\tau\,\|\boldsymbol{O}_{\theta}(\tau)\ \boldsymbol{\Phi}_{\theta}^{-1}(t)\|_1\leq\\&\leq\|\boldsymbol{I}+\tau\boldsymbol{M}_{\theta}(t)+\tau\boldsymbol{O}_{\theta}(\tau)\ \boldsymbol{\Phi}_{\theta}^{-1}(t)\|_1\leq\\&\leq\|\boldsymbol{I}+\tau\boldsymbol{M}_{\theta}(t)\|_1+\tau\|\boldsymbol{O}_{\theta}(\tau)\ \boldsymbol{\Phi}_{\theta}^{-1}(t)\|_1,\end{aligned}$$

and we can write

$$\begin{aligned}&\frac{\|\boldsymbol{I}+\tau\boldsymbol{M}_{\theta}(t)\|_1-1}{\tau}-\|\boldsymbol{O}_{\theta}(\tau)\boldsymbol{\Phi}_{\theta}^{-1}(t)\|_1\leq\frac{\|\boldsymbol{\Psi}_{\theta}(t+\tau,t)\|_1-1}{\tau}\leq\\&\leq\frac{\|\boldsymbol{I}+\tau\boldsymbol{M}_{\theta}(t)\|_1-1}{\tau}+\|\boldsymbol{O}_{\theta}(\tau)\boldsymbol{\Phi}_{\theta}^{-1}(t)\|_1,\end{aligned}$$

yielding

$$\lim_{\tau\downarrow 0}\frac{\|\boldsymbol{\Psi}_{\theta}(t+\tau,t)\|_1-1}{\tau}=\mu_1(\boldsymbol{M}_{\theta}(t)),\quad \theta=1,\ldots,N,\quad t\in\mathbb{R}_+. \quad \text{(14-CT)}$$

Let $\boldsymbol{y}$ solve a subsystem (12-CT) and let $t,t_0\in\mathbb{R}_+$, $t\geq t_0$. Set $\varepsilon=\|\boldsymbol{y}(t_0)\|_1$. If $\varepsilon>0$, then $\overline{\boldsymbol{y}}=\varepsilon^{-1}\boldsymbol{y}$ also solves (12-CT) and satisfies $\|\overline{\boldsymbol{y}}(t_0)\|_1=1$. Since the set $\|\boldsymbol{y}\|_1\leq 1$ is positively invariant with respect to the considered subsystem (12-CT), we have $\|\overline{\boldsymbol{y}}(t)\|_1\leq 1$, or, equivalently $\|\varepsilon^{-1}\boldsymbol{y}(t)\|_1\leq 1$. Thus, for any $t,t_0\in\mathbb{R}_+$, $t\geq t_0$, $\|\boldsymbol{y}(t)\|_1\leq\varepsilon=\|\boldsymbol{y}(t_0)\|_1$, i.e the function $\|\boldsymbol{y}(t)\|_1$ is non-increasing along each trajectory of subsystem (12-CT).

On the other hand, for arbitrary $t,t_0\in\mathbb{R}_+$, $t\geq t_0$, and arbitrary $\theta=1,\ldots,N$, there exists a vector $\boldsymbol{y}_0\in\mathbb{R}^n$, $\|\boldsymbol{y}_0\|_1=1$, such that $\|\boldsymbol{\Psi}_{\theta}(t,t_0)\|_1=\|\boldsymbol{\Psi}_{\theta}(t,t_0)\boldsymbol{y}_0\|_1$. If $\boldsymbol{y}(t_0)=\boldsymbol{y}_0$, then for $\boldsymbol{y}(t)=\boldsymbol{\Psi}_{\theta}(t,t_0)\boldsymbol{y}(t_0)$ we have $\|\boldsymbol{y}(t)\|_1\leq\|\boldsymbol{y}(t_0)\|_1=1$, due to the non-increasing monotonicity of the function $\|\boldsymbol{y}(t)\|_1$ along any trajectory of subsystem (12-CT). Hence $\|\boldsymbol{\Psi}_{\theta}(t,t_0)\|_1=\|\boldsymbol{y}(t)\|_1\leq 1$, yielding $\| \boldsymbol{\Psi}_\theta(t+\tau, t) \|_1 \le 1$ for any $t \in \mathbb{R}_+$ and $\tau \ge 0$. Inequalities (14-CT) show that $\mu_1(\boldsymbol{M}_\theta(t)) \le 0$, meaning that inequalities (13-CT) are true. By using $\boldsymbol{\Gamma}(t)$ as defined in (10-CT), inequalities (13-CT) are further equivalent to

$$(\boldsymbol{A}_\theta^{\mathrm{CT}})^T \boldsymbol{\gamma}(t) \le -\dot{\boldsymbol{\gamma}}(t), t \in \mathbb{R}_+, \theta = 1, \ldots, N, \quad \boldsymbol{\gamma} : \mathbb{R}_+ \to \mathbb{R}^n, \boldsymbol{\gamma}(t) = [\gamma_1(t) \ldots \gamma_n(t)]^T \tag{15-CT}$$

which show that the inequalities

$$(\underline{\boldsymbol{A}}_{\boldsymbol{\sigma}}^{\mathrm{CT}})^T \boldsymbol{\gamma}(t) \le -\dot{\boldsymbol{\gamma}}(t),\ t \in \mathbb{R}_+ \Leftrightarrow -(\underline{\boldsymbol{A}}_{\boldsymbol{\sigma}}^{\mathrm{CT}})^T \boldsymbol{\gamma}(t) \ge \dot{\boldsymbol{\gamma}}(t),\ t \in \mathbb{R}_+ \Leftrightarrow e^{-(\underline{\boldsymbol{A}}_{\boldsymbol{\sigma}}^{\mathrm{CT}})^T t} \boldsymbol{\gamma}(0) \ge \boldsymbol{\gamma}(t),\ t \in \mathbb{R}_+,\ \boldsymbol{\sigma} \in \mathcal{C}, \tag{16-CT}$$

hold true for all column representatives $\underline{\boldsymbol{A}}_{\boldsymbol{\sigma}}^{\mathrm{CT}}$, $\boldsymbol{\sigma} \in \mathcal{C}$, corresponding to the set of essentially nonnegative matrices $\mathcal{A}^{\mathrm{CT}}$ (11-CT). Any column representative $\underline{\boldsymbol{A}}_{\boldsymbol{\sigma}}^{\mathrm{CT}}, \boldsymbol{\sigma} \in \mathcal{C}$ is also an essentially nonnegative matrix and $e^{(\underline{\boldsymbol{A}}_{\boldsymbol{\sigma}}^{\mathrm{CT}})^T t}$ is nonnegative, so that the left multiplication of the third form of inequalities (16-CT) yields

$$\boldsymbol{\gamma}(0) \ge e^{(\underline{\boldsymbol{A}}_{\boldsymbol{\sigma}}^{\mathrm{CT}})^T t} \boldsymbol{\gamma}(t),\ t \in \mathbb{R}_+,\ \boldsymbol{\sigma} \in \mathcal{C}.$$

Since $\lim_{t \to \infty} \gamma_i(t) = \infty$, $i = 1, \ldots, n$, we can find a moment $\underline{t} \in \mathbb{R}_+$ such that $\boldsymbol{\gamma}(0) \le \varphi \boldsymbol{\gamma}(\underline{t})$, $0 < \varphi < 1$, which, together with $e^{(\underline{\boldsymbol{A}}_{\boldsymbol{\sigma}}^{\mathrm{CT}})^T \underline{t}} \boldsymbol{\gamma}(\underline{t}) \le \boldsymbol{\gamma}(0)$ allow us to write the inequalities

$$e^{(\underline{\boldsymbol{A}}_{\boldsymbol{\sigma}}^{\mathrm{CT}})^T \underline{t}} \boldsymbol{\gamma}(\underline{t}) \le \varphi \boldsymbol{\gamma}(\underline{t}),\ \gamma(\underline{t}) \gg 0, \quad 0 < \varphi < 1, \boldsymbol{\sigma} \in \mathcal{C}. \tag{17-CT}$$

For any $\boldsymbol{\sigma} \in \mathcal{C}$, the Perron-Frobenius eigenvalue of the non-negative matrix $e^{(\underline{\boldsymbol{A}}_{\boldsymbol{\sigma}}^{\mathrm{CT}})^T \underline{t}}$ fulfills the inequality (as per [11, Corollary 8.1.29])

$$\lambda_{PF}\left(e^{(\underline{\boldsymbol{A}}_{\boldsymbol{\sigma}}^{\mathrm{CT}})^T \underline{t}}\right) \le \varphi < 1, \quad \boldsymbol{\sigma} \in \mathcal{C} \tag{18-CT}$$

proving that $\lambda_{PF}\left(\underline{\boldsymbol{A}}_{\boldsymbol{\sigma}}^{\mathrm{CT}}\right) < 0$, $\boldsymbol{\sigma} \in \mathcal{C}$. Equivalently all column representatives $\underline{\boldsymbol{A}}_{\boldsymbol{\sigma}}^{\mathrm{CT}}$, $\boldsymbol{\sigma} \in \mathcal{C}$, of the essentially non-negative matrices $\mathcal{A}^{\mathrm{CT}}$ (11-CT) are Hurwitz stable.

Sufficiency: The switching *positive* system (1-CT) with *essentially non-negative* constituent matrices $\mathcal{A}^{\mathrm{CT}}$ (11-CT) has invariant sets of form $X^{\mathrm{CT}}(t)$ (9-CT) as per the implication (S1)⇒(S4) discussed in Subsection 1.2. This means that the switching positive system (1-CT) with the constituent matrices $\mathcal{A}^{\mathrm{CT}}$ (11-CT) can serve as a *comparison system* (e.g. see [12]) for the switching system (1-CT) with *arbitrary* constituent matrices $\mathcal{A}$ (2). Therefore, the nonnegative set $X^{\mathrm{CT}}(t)$ (9-CT) is invariant with respect to the 1-norm of the trajectories of switching system (1-CT) with arbitrary constituent matrices $\mathcal{A}$ (2). Equivalently, the sets $\left\{\boldsymbol{x} \in \mathbb{R}^n \mid \| \mathrm{diag}\{\boldsymbol{u}\} \boldsymbol{x} \|_1 \le e^{rt}\right\}$, $t \in \mathbb{R}_+$, $\boldsymbol{u} \gg 0$, $r < 0$, are invariant with respect to the trajectories of switching system (1-CT) with arbitrary constituent matrices $\mathcal{A}$ (2). These invariant sets can also be described as $\left\{\boldsymbol{x} \in \mathbb{R}^n \mid \| e^{-rt} \mathrm{diag}\{\boldsymbol{u}\} \boldsymbol{x} \|_1 \le 1\right\}$. In other words, we have proven the existence of a set of form $X_{\boldsymbol{\Gamma}}^{\mathrm{CT}}(t)$ (10-CT), with $\gamma_i(t) = u_i e^{-rt}$, $u_i > 0$, $i = 1, \ldots, n$, $r < 0$, which is invariant with respect to switching system (1-CT) with *arbitrary* constituent matrices $\mathcal{A}$ (2). ■

Remark 1-CT

The proof of Sufficiency of Theorem 1-CT uses the simplest form for the invariant sets $X_{\boldsymbol{\Gamma}}^{\mathrm{CT}}(t)$ (10-CT). It is worth sketching a procedure for constructing invariant sets with a more general form. If all column representatives $\underline{\boldsymbol{A}}_{\boldsymbol{\sigma}}^{\mathrm{CT}}$, $\boldsymbol{\sigma} \in \mathcal{C}$, of the essentially non-negative matrices $\mathcal{A}^{\mathrm{CT}}$ (11-CT) are Hurwitz stable, then, for the quasi–linear weak inequalities (7-CT) written for $\boldsymbol{A}_\theta^{\mathrm{CT}}$, we can find several solutions, as per implication (S1)⇒(S2) discussed in Subsection 1.2. For instance, consider the vectors $\boldsymbol{u}_k \gg 0$, and the constants $r_k < 0$, $k = 1, \ldots, K$, which solve inequalities (7-CT) written for $\boldsymbol{A}_\theta^{\mathrm{CT}}$, and define the vector function

$$\boldsymbol{\gamma}(t) = \sum_{k=1}^{K} c_k \boldsymbol{u}_k e^{-r_k t},\ c_k > 0, k = 1, \ldots, K. \tag{19-CT}$$

The function $\boldsymbol{\gamma}(t)$ (19-CT) is a solution to inequalities (15-CT) and $\boldsymbol{\Gamma}(t) = \mathrm{diag}\{\boldsymbol{\gamma}(t)\}$ satisfies inequalities (13-CT). Equivalently, the set $X_{\boldsymbol{\Gamma}}^{\mathrm{CT}}(t)$ defined by (10-CT) with $\boldsymbol{\Gamma}(t) = \mathrm{diag}\{\sum_{k=1}^{K} c_k \boldsymbol{u}_k e^{-r_k t}\}$ is invariant with respect to the trajectories of the switching system (1-CT) with *arbitrary* constituent matrices $\mathcal{A}$ (2). ■

Remark 2-CT

Theorem 1-CT includes the equivalence (S1)⇔(S4) discussed in Subsection 1.2 as a particular case. Indeed, Theorem 1-CT can be applied to a switching *positive* system (1-CT) with $\underline{\boldsymbol{A}}_{\boldsymbol{\sigma}}^{\mathrm{CT}} = \underline{\boldsymbol{A}}_{\boldsymbol{\sigma}}$, $\boldsymbol{\sigma} \in \mathcal{C}$, and $\boldsymbol{\Gamma}(t) = \mathrm{diag}\{e^{-rt} \boldsymbol{u}\}$, $\boldsymbol{u} \gg 0$, $r < 0$. ■

V. Example

Consider an arbitrary switching linear system in continuous-time defined by the matrices from the set $\mathcal{A} = \{\boldsymbol{A}_1, \boldsymbol{A}_2\} \subset \mathbb{R}^{3\times 3}$, with

$$\boldsymbol{A}_1 = \begin{bmatrix} -1 & 0 & 0 \\ -1/16 & -1 & 1 \\ -1/100 & 1/10 & -1 \end{bmatrix}, \boldsymbol{A}_2 = \begin{bmatrix} -1 & 0 & 0 \\ 1/100 & -1 & -1 \\ 1/100 & -1/100 & -1 \end{bmatrix}, \tag{20}$$

inspired by [7]. In order to apply Theorem 1-CT we build the set of essentially non-negative matrices $\mathcal{A}^{\mathrm{CT}} = \left\{\boldsymbol{A}_1^{\mathrm{CT}}, \boldsymbol{A}_2^{\mathrm{CT}}\right\}$, given by (11-CT). The dominant eigenvalue of all the column representatives corresponding to the set $\mathcal{A}^{\mathrm{CT}}$ is $\underline{\lambda}^* = \max_{\boldsymbol{\sigma} \in \mathcal{C}} \lambda_{\max}(\underline{\boldsymbol{A}}_{\boldsymbol{\sigma}}^{\mathrm{CT}}) = -0.6838 < 0$, therefore all these column representatives are Hurwitz

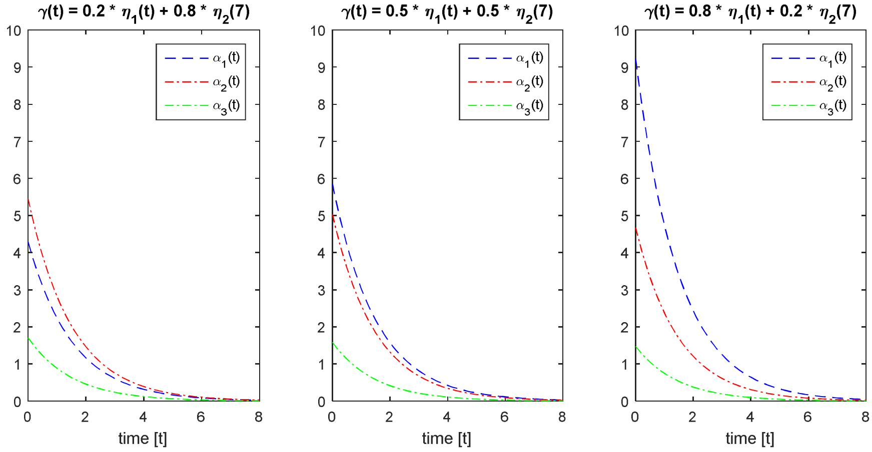

Figure 1. Time dependence of the hyper-axes of the hyper-rhomb defined by $\gamma(t)=c_1\eta_1(t)+c_2\eta_2(t)$ for (a) $c_1=0.2 \quad c_2=0.8$, (b) $c_1=0.5 \quad c_2=0.5$, and (c) $c_1=0.8 \quad c_2=0.2$.

stable. Consequently, Theorem 1-CT implies that there exist sets of form (10-CT) that are invariant with respect to arbitrary switching system (1-CT)&(20).

Indeed, it is easy to test that the vector functions $\eta_1(t)=[0.067 \;\; 0.224 \;\; 0.709]^T \cdot e^{0.683t}$ and $\eta_2(t)=[0.274 \;\; 0.173 \;\; 0.553]^T \cdot e^{0.680t}$ satisfy the differential inequalities (15-CT). Moreover, any convex combination $\gamma(t)=c_1\eta_1(t)+c_2\eta_2(t)=[\gamma_1(t),\gamma_1(t),\gamma_3(t)]^T$, with $c_1,c_2>0$, $c_1+c_2=1$, also satisfies (15-CT) and can be used to define a time-dependent set of form (10-CT) $X_{\mathbf{\Gamma}}^{\mathrm{CT}}(t)=\{\boldsymbol{x}\in\mathbb{R}^3 \mid \|\mathbf{\Gamma}(t)\boldsymbol{x}\|_1\le 1\}$, hyper-rhomb with $\mathbf{\Gamma}(t)=\mathrm{diag}\{\boldsymbol{\gamma}(t)\}$, that is flow invariant with respect to the trajectories of the arbitrary switching system (1-CT)&(20). The set $X_{\mathbf{\Gamma}}^{\mathrm{CT}}(t)$ is a hyper-rhomb; the time-dependence of the hyper-axes $\alpha_i(t)=1/\gamma_i(t)$, $i=1,2,3$, is presented in Figure 1 for the combinations (a) $c_1=0.2 \quad c_2=0.8$, (b) $c_1=0.5 \quad c_2=0.5$, and (c) $c_1=0.8 \quad c_2=0.2$.

VI. Conclusions

The paper enlarges the mathematical framework that exploits column representatives as an instrument for the qualitative analysis of the arbitrary switching linear systems. The expansion refers to the connections between the algebraic properties of the column representatives and the invariance properties of the switching system trajectories. Thus, the paper proves the equivalence between: • the Schur (Hurwitz respectively) stability of the column representatives corresponding to a matrix set adequately built from the original system matrices, • the existence of sets with general form time-dependence, which are invariant with respect to the switching system trajectories. This equivalence includes, as particular cases, several results previously reported by separate papers on switching systems with positive dynamics. The considered case study offers a numerical illustration for the theoretical developments corresponding to the continuous-time case.

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